

# ON $p$ -ADIC ANALOGUE OF $q$ -BERNSTEIN POLYNOMIALS AND RELATED INTEGRALS

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**Abstract** Recently, T. Kim([5]) introduced  $q$ -Bernstein polynomials which are different  $q$ -Bernstein polynomials of Phillips  $q$ -Bernstein polynomials([11, 12]). The purpose of this paper is to study some properties of several type Kim's  $q$ -Bernstein polynomials to express the  $p$ -adic  $q$ -integral of these polynomials on  $\mathbb{Z}_p$  associated with Carlitz's  $q$ -Bernoulli numbers and polynomials. Finally, we also derive some relations on the  $p$ -adic  $q$ -integral of the products of several type Kim's  $q$ -Bernstein polynomials and the powers of them on  $\mathbb{Z}_p$ .

## 1. INTRODUCTION

Let  $C[0, 1]$  denote the set of continuous functions on  $[0, 1]$ . For  $0 < q < 1$  and  $f \in C[0, 1]$ , Kim introduced the  $q$ -extension of Bernstein linear operator of order  $n$  for  $f$  as follows:

$$\mathbb{B}_{n,q}(f|x) = \sum_{k=0}^n f\left(\frac{k}{n}\right) \binom{n}{k} [x]_q^k [1-x]_{\frac{1}{q}}^{n-k} = \sum_{k=0}^n f\left(\frac{k}{n}\right) B_{k,n}(x, q),$$

where  $[x]_q = \frac{1-q^x}{1-q}$ , (see [5]). Here  $\mathbb{B}_{n,q}(f|x)$  is called Kim's  $q$ -Bernstein operator of order  $n$  for  $f$ . For  $k, n \in \mathbb{Z}_+ (= \mathbb{N} \cup \{0\})$ ,  $B_{k,n}(x, q) = \binom{n}{k} [x]_q^k [1-x]_{\frac{1}{q}}^{n-k}$  are called the Kim's  $q$ -Bernstein polynomials of degree  $n$  (see [1, 6, 11-13]).

In [2], Carlitz defined a set of numbers  $\xi_k = \xi_k(q)$  inductively by

$$\xi_0 = 1, \quad (q\xi + 1)^k - \xi_k = \begin{cases} 1 & \text{if } k = 1, \\ 0 & \text{if } k > 1, \end{cases}$$

with the usual convention of replacing  $\xi^k$  by  $\xi_k$ . These numbers are  $q$ -analogues of ordinary Bernoulli numbers  $B_k$ , but they do not remain finite for  $q = 1$ . So he modified the definition as follows:

$$\beta_{0,q} = 1, \quad q(q\beta + 1)^k - \beta_{k,q} = \begin{cases} 1 & \text{if } k = 1, \\ 0 & \text{if } k > 1, \end{cases}$$

with the usual convention of replacing  $\beta^k$  by  $\beta_{k,q}$  (see [2]). These numbers  $\beta_{n,q}$  are called the  $n$ -th Carlitz  $q$ -Bernoulli numbers. And Carlitz's  $q$ -Bernoulli polynomials are defined by

$$\beta_{k,q}(x) = (q^x \beta + [x]_q)^k = \sum_{i=0}^k \binom{k}{i} \beta_{i,q} q^{ix} [x]_q^{k-i}.$$

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As  $q \rightarrow 1$ , we have  $\beta_{k,q} \rightarrow B_k$  and  $\beta_{k,q}(x) \rightarrow B_k(x)$ , where  $B_k$  and  $B_k(x)$  are the ordinary Bernoulli numbers and polynomials, respectively.

Let  $p$  be a fixed prime number. Throughout this paper,  $\mathbb{Z}$ ,  $\mathbb{Q}$ ,  $\mathbb{Z}_p$ ,  $\mathbb{Q}_p$  and  $\mathbb{C}_p$  will denote the ring of rational integers, the field of rational numbers, the ring of  $p$ -adic integers, the field of  $p$ -adic rational numbers and the completion of algebraic closure of  $\mathbb{Q}_p$ , respectively. Let  $\nu_p$  be the normalized exponential valuation of  $\mathbb{C}_p$  such that  $|p|_p = p^{-\nu_p(p)} = \frac{1}{p}$ .

Let  $q$  be regarded as either a complex number  $q \in \mathbb{C}$  or a  $p$ -adic number  $q \in \mathbb{C}_p$ . If  $q \in \mathbb{C}$ , we assume  $|q| < 1$ , and if  $q \in \mathbb{C}_p$ , we normally assume  $|1 - q|_p < 1$ .

We say that  $f$  is a uniformly differentiable function at a point  $a \in \mathbb{Z}_p$  and denote this property by  $f \in UD(\mathbb{Z}_p)$  if the difference quotient  $F_f(x, y) = \frac{f(x) - f(y)}{x - y}$  has a limit  $f'(a)$  as  $(x, y) \rightarrow (a, a)$  (see [3-10]).

For  $f \in UD(\mathbb{Z}_p)$ , let us begin with the expression

$$\frac{1}{[p^N]_q} \sum_{0 \leq x < p^N} q^x f(x) = \sum_{0 \leq x < p^N} f(x) \mu_q(x + p^N \mathbb{Z}_p), \quad (1)$$

representing a  $q$ -analogue of the Riemann sums for  $f$  (see [8]). The integral of  $f$  on  $\mathbb{Z}_p$  is defined as the limit as  $n \rightarrow \infty$  of the sums (if exists). The  $p$ -adic  $q$ -integral on a function  $f \in UD(\mathbb{Z}_p)$  is defined by

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \rightarrow \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x, \quad (\text{see [8]}).$$

As was shown in [6], Carlitz's  $q$ -Bernoulli numbers can be represented by  $p$ -adic  $q$ -integral on  $\mathbb{Z}_p$  as follows:

$$\int_{\mathbb{Z}_p} [x]_q^m d\mu_q(x) = \beta_{m,q}, \quad \text{for } m \in \mathbb{Z}_+. \quad (2)$$

Also, Carlitz's  $q$ -Bernoulli polynomials  $\beta_{k,q}(x)$  can be represented

$$\beta_{m,q}(x) = \int_{\mathbb{Z}_p} [x + y]_q^m d\mu_q(y), \quad \text{for } m \in \mathbb{Z}_+, \quad (\text{see [6]}). \quad (3)$$

In this paper, we consider the  $p$ -adic analogue of Kim's  $q$ -Bernstein polynomials on  $\mathbb{Z}_p$  and give some properties of the several type Kim's  $q$ -Bernstein polynomials to represent the  $p$ -adic  $q$ -integral on  $\mathbb{Z}_p$  of these polynomials. Finally, we derive some relations on the  $p$ -adic  $q$ -integral of the products of several type Kim's  $q$ -Bernstein polynomials and the powers of them on  $\mathbb{Z}_p$ .

## 2. $q$ -BERNSTEIN POLYNOMIALS ASSOCIATED WITH $p$ -ADIC $q$ -INTEGRAL ON $\mathbb{Z}_p$

In this section, we assume that  $q \in \mathbb{C}_p$  with  $|1 - q|_p < 1$ .

From (1), (2) and (3), we note that

$$\int_{\mathbb{Z}_p} [1 - x + x_1]_q^n d\mu_{\frac{1}{q}}(x_1) = \frac{q^n}{(q - 1)^{n-1}} \sum_{l=0}^n \binom{n}{l} (-1)^l q^{lx} \frac{l+1}{q^{l+1} - 1}, \quad (4)$$

and

$$\int_{\mathbb{Z}_p} [x + x_1]_q^n d\mu_q(x_1) = \frac{1}{(q - 1)^{n-1}} \sum_{l=0}^n \binom{n}{l} (-1)^l q^{lx} \frac{l+1}{1 - q^{l+1}}. \quad (5)$$

By (4) and (5), we get

$$(-1)^n q^n \int_{\mathbb{Z}_p} [x + x_1]_q^n d\mu_q(x_1) = \int_{\mathbb{Z}_p} [1 - x + x_1]_{\frac{1}{q}}^n d\mu_{\frac{1}{q}}(x_1).$$

Therefore, we obtain the following theorem.

**Theorem 1.** *For  $n \in \mathbb{Z}_+$ , we have*

$$\int_{\mathbb{Z}_p} [1 - x + x_1]_{\frac{1}{q}}^n d\mu_{\frac{1}{q}}(x_1) = (-1)^n q^n \int_{\mathbb{Z}_p} [x + x_1]_q^n d\mu_q(x_1).$$

By the definition of Carlitz's  $q$ -Bernoulli numbers and polynomials, we get

$$q^2 \beta_{n,q}(2) - (n+1)q^2 + q = q(q\beta + 1)^n = \beta_{n,q}, \quad \text{if } n > 1.$$

Thus, we have the following proposition.

**Proposition 2.** *For  $n \in \mathbb{N}$  with  $n > 1$ , we have*

$$\beta_{n,q}(2) = \frac{1}{q^2} \beta_{n,q} + n + 1 - \frac{1}{q}.$$

It is easy to show that

$$[1 - x]_{\frac{1}{q}}^n = (1 - [x]_q)^n = (-1)^n q^n [x - 1]_q^n.$$

Hence, we have

$$\int_{\mathbb{Z}_p} [1 - x]_{\frac{1}{q}}^n d\mu_q(x) = (-1)^n q^n \int_{\mathbb{Z}_p} [x - 1]_q^n d\mu_q(x).$$

By (3), we get

$$\int_{\mathbb{Z}_p} [1 - x]_{\frac{1}{q}}^n d\mu_q(x) = (-1)^n q^n \beta_{n,q}(-1). \quad (6)$$

By Theorem 1 and (6), we see that

$$\int_{\mathbb{Z}_p} [1 - x]_{\frac{1}{q}}^n d\mu_q(x) = (-1)^n q^n \beta_{n,q}(-1) = \beta_{n,\frac{1}{q}}(2). \quad (7)$$

From (7) and Proposition 2, we have

$$\int_{\mathbb{Z}_p} [1 - x]_{\frac{1}{q}}^n d\mu_q(x) = \beta_{n,\frac{1}{q}}(2) = q^2 \beta_{n,\frac{1}{q}} + n + 1 - q. \quad (8)$$

By (2) and (8), we obtain the following theorem.

**Theorem 3.** *For  $n \in \mathbb{N}$  with  $n > 1$ , we have*

$$\int_{\mathbb{Z}_p} [1 - x]_{\frac{1}{q}}^n d\mu_q(x) = q^2 \int_{\mathbb{Z}_p} [x]_{\frac{1}{q}}^n d\mu_{\frac{1}{q}}(x) + n + 1 - q.$$

Taking the  $p$ -adic  $q$ -integral on  $\mathbb{Z}_p$  for one Kim's  $q$ -Bernstein polynomials, we get

$$\begin{aligned} \int_{\mathbb{Z}_p} B_{k,n}(x, q) d\mu_q(x) &= \binom{n}{k} \int_{\mathbb{Z}_p} [x]_q^k [1-x]_{\frac{1}{q}}^{n-k} d\mu_q(x) \\ &= \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \int_{\mathbb{Z}_p} [x]_q^{k+l} d\mu_q(x) \\ &= \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \beta_{k+l, q}, \end{aligned} \quad (9)$$

and, by the  $q$ -symmetric property of  $B_{k,n}(x, q)$ , we see that

$$\begin{aligned} \int_{\mathbb{Z}_p} B_{k,n}(x, q) d\mu_q(x) &= \int_{\mathbb{Z}_p} B_{n-k,n}(1-x, \frac{1}{q}) d\mu_q(x) \\ &= \binom{n}{k} \sum_{l=0}^k \binom{k}{l} (-1)^{k+l} \int_{\mathbb{Z}_p} [1-x]_{\frac{1}{q}}^{n-l} d\mu_q(x). \end{aligned} \quad (10)$$

For  $n > k + 1$ , by Theorem 3 and (10), we have

$$\begin{aligned} \int_{\mathbb{Z}_p} B_{k,n}(x, q) d\mu_q(x) &= \binom{n}{k} \sum_{l=0}^k (-1)^{k+l} \binom{k}{l} [n-l+1-q+q^2 \int_{\mathbb{Z}_p} [x]_{\frac{1}{q}}^{n-l} d\mu_{\frac{1}{q}}(x)] \\ &= \binom{n}{k} \sum_{l=0}^k (-1)^{k+l} \binom{k}{l} [n-l+1-q+q^2 \beta_{n-l, \frac{1}{q}}]. \end{aligned} \quad (11)$$

Let  $m, n, k \in \mathbb{Z}_+$  with  $m+n > 2k+1$ . Then the  $p$ -adic  $q$ -integral for the multiplication of two Kim's  $q$ -Bernstein polynomials on  $\mathbb{Z}_p$  can be given by the following relation:

$$\begin{aligned} \int_{\mathbb{Z}_p} B_{k,n}(x, q) B_{k,m}(x, q) d\mu_q(x) &= \binom{n}{k} \binom{m}{k} \int_{\mathbb{Z}_p} [x]_q^{2k} [1-x]_{\frac{1}{q}}^{n+m-2k} d\mu_q(x) \\ &= \binom{n}{k} \binom{m}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{l+2k} \int_{\mathbb{Z}_p} [1-x]_{\frac{1}{q}}^{n+m-l} d\mu_q(x). \end{aligned} \quad (12)$$

By Theorem 3 and (12), we get

$$\begin{aligned} \int_{\mathbb{Z}_p} B_{k,n}(x, q) B_{k,m}(x, q) d\mu_q(x) &= \binom{n}{k} \binom{m}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{l+2k} [n+m-l+1-q+q^2 \int_{\mathbb{Z}_p} [x]_{\frac{1}{q}}^{n+m-l} d\mu_{\frac{1}{q}}(x)] \\ &= \binom{n}{k} \binom{m}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{l+2k} [n+m-l+1-q+q^2 \beta_{n+m-l, \frac{1}{q}}]. \end{aligned} \quad (13)$$

By the simple calculation, we easily get

$$\begin{aligned}
& \int_{\mathbb{Z}_p} B_{k,n}(x, q) B_{k,m}(x, q) d\mu_q(x) \\
&= \binom{n}{k} \binom{m}{k} \int_{\mathbb{Z}_p} [x]_q^{2k} [1-x]_{\frac{1}{q}}^{n+m-2k} d\mu_q(x) \\
&= \binom{n}{k} \binom{m}{k} \sum_{l=0}^{n+m-2k} \binom{n+m-2k}{l} (-1)^l \int_{\mathbb{Z}_p} [x]_q^{l+2k} d\mu_q(x) \\
&= \binom{n}{k} \binom{m}{k} \sum_{l=0}^{n+m-2k} \binom{n+m-2k}{l} (-1)^l \beta_{l+2k, q}.
\end{aligned} \tag{14}$$

Continuing this process, we obtain

$$\begin{aligned}
& \int_{\mathbb{Z}_p} \left( \prod_{i=1}^s B_{k,n_i}(x, q) \right) d\mu_q(x) \\
&= \left( \prod_{i=1}^s \binom{n_i}{k} \right) \int_{\mathbb{Z}_p} [x]_q^{sk} [1-x]_{\frac{1}{q}}^{n_1+\dots+n_s-sk} d\mu_q(x) \\
&= \left( \prod_{i=1}^s \binom{n_i}{k} \right) \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{sk+l} \int_{\mathbb{Z}_p} [1-x]_{\frac{1}{q}}^{n_1+\dots+n_s-l} d\mu_q(x).
\end{aligned} \tag{15}$$

Let  $s \in \mathbb{N}$  and  $n_1, \dots, n_s, k \in \mathbb{Z}_+$  with  $n_1 + n_2 + \dots + n_s > sk + 1$ . By Theorem 3 and (15), we get

$$\begin{aligned}
& \int_{\mathbb{Z}_p} \left( \prod_{i=1}^s B_{k,n_i}(x, q) \right) d\mu_q(x) \\
&= \left( \prod_{i=1}^s \binom{n_i}{k} \right) \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{sk+l} \left\{ \sum_{i=1}^s n_i - l + 1 - q + q^2 \int_{\mathbb{Z}_p} [x]_{\frac{1}{q}}^{n_1+\dots+n_s-l} d\mu_{\frac{1}{q}}(x) \right\} \\
&= \left( \prod_{i=1}^s \binom{n_i}{k} \right) \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{sk+l} \left\{ \sum_{i=1}^s n_i - l + 1 - q + q^2 \beta_{n_1+\dots+n_s-l, \frac{1}{q}} \right\}.
\end{aligned} \tag{16}$$

From the definition of binomial coefficient, we note that

$$\begin{aligned}
& \int_{\mathbb{Z}_p} \left( \prod_{i=1}^s B_{k,n_i}(x, q) \right) d\mu_q(x) \\
&= \left( \prod_{i=1}^s \binom{n_i}{k} \right) \int_{\mathbb{Z}_p} [x]_q^{sk} [1-x]_{\frac{1}{q}}^{n_1+\dots+n_s-sk} d\mu_q(x) \\
&= \left( \prod_{i=1}^s \binom{n_i}{k} \right) \sum_{l=0}^{n_1+\dots+n_s-sk} \binom{n_1+\dots+n_s-sk}{l} (-1)^l \int_{\mathbb{Z}_p} [x]_q^{sk+l} d\mu_q(x) \\
&= \left( \prod_{i=1}^s \binom{n_i}{k} \right) \sum_{l=0}^{n_1+\dots+n_s-sk} \binom{n_1+\dots+n_s-sk}{l} (-1)^l \beta_{sk+l, q},
\end{aligned} \tag{17}$$

where  $s \in \mathbb{N}$  and  $n_1, \dots, n_s, k \in \mathbb{Z}_+$ .

By (16) and (17), we obtain the following theorem.

**Theorem 4.** (I) For  $s \in \mathbb{N}$  and  $n_1, \dots, n_s, k \in \mathbb{N}$  with  $n_1 + n_2 + \dots + n_s > sk + 1$ , we have

$$\begin{aligned} & \int_{\mathbb{Z}_p} \left( \prod_{i=1}^s B_{k, n_i}(x, q) \right) d\mu_q(x) \\ &= \left( \prod_{i=1}^s \binom{n_i}{k} \right) \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{sk+l} \left\{ \sum_{i=1}^s n_i - l + 1 - q + q^2 \beta_{n_1 + \dots + n_s - l, \frac{1}{q}} \right\}. \end{aligned}$$

(II) For  $s \in \mathbb{N}$  and  $n_1, \dots, n_s, k \in \mathbb{Z}_+$ , we have

$$\begin{aligned} & \int_{\mathbb{Z}_p} \left( \prod_{i=1}^s B_{k, n_i}(x, q) \right) d\mu_q(x) \\ &= \left( \prod_{i=1}^s \binom{n_i}{k} \right) \sum_{l=0}^{n_1 + \dots + n_s - sk} \binom{n_1 + \dots + n_s - sk}{l} (-1)^l \beta_{sk+l, q}. \end{aligned}$$

By Theorem 4, we obtain the following corollary.

**Corollary 5.** For  $s \in \mathbb{N}$  and  $n_1, \dots, n_s, k \in \mathbb{N}$  with  $n_1 + n_2 + \dots + n_s > sk + 1$ , we have

$$\begin{aligned} & \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{sk+l} \left\{ \sum_{i=1}^s n_i - l + 1 - q + q^2 \beta_{n_1 + \dots + n_s - l, \frac{1}{q}} \right\} \\ &= \sum_{l=0}^{n_1 + \dots + n_s - sk} \binom{n_1 + \dots + n_s - sk}{l} (-1)^l \beta_{sk+l, q}. \end{aligned}$$

Let  $s \in \mathbb{N}$  and  $m_1, \dots, m_s, n_1, \dots, n_s, k \in \mathbb{Z}_+$  with  $m_1 n_1 + \dots + m_s n_s > (m_1 + \dots + m_s)k + 1$ . Then we have

$$\begin{aligned} & \int_{\mathbb{Z}_p} \left( \prod_{i=1}^s B_{k, n_i}^{m_i}(x, q) \right) d\mu_q(x) \tag{18} \\ &= \left( \prod_{i=1}^s \binom{n_i}{k}^{m_i} \right) \sum_{l=0}^{k \sum_{i=1}^s m_i} \binom{k \sum_{i=1}^s m_i}{l} (-1)^{k \sum_{i=1}^s m_i - l} \\ & \quad \times \int_{\mathbb{Z}_p} [1 - x]_q^{\sum_{i=1}^s n_i m_i - l} d\mu_q(x) \\ &= \left( \prod_{i=1}^s \binom{n_i}{k}^{m_i} \right) \sum_{l=0}^{k \sum_{i=1}^s m_i} \binom{k \sum_{i=1}^s m_i}{l} (-1)^{k \sum_{i=1}^s m_i - l} \\ & \quad \times \left\{ \left( \sum_{i=1}^s m_i n_i - l + 1 \right) - q + q^2 \int_{\mathbb{Z}_p} [x]_{\frac{1}{q}}^{\sum_{i=1}^s n_i m_i - l} d\mu_{\frac{1}{q}}(x) \right\} \\ &= \left( \prod_{i=1}^s \binom{n_i}{k}^{m_i} \right) \sum_{l=0}^{k \sum_{i=1}^s m_i} \binom{k \sum_{i=1}^s m_i}{l} (-1)^{k \sum_{i=1}^s m_i - l} \\ & \quad \times \left\{ \left( \sum_{i=1}^s m_i n_i - l + 1 \right) - q + q^2 \beta_{n_1 m_1 + \dots + n_s m_s - l, \frac{1}{q}} \right\}. \end{aligned}$$

From the definition of binomial coefficient, we have

$$\begin{aligned}
& \int_{\mathbb{Z}_p} \left( \prod_{i=1}^s B_{k, n_i}^{m_i}(x, q) \right) d\mu_q(x) \\
&= \left( \prod_{i=1}^s \binom{n_i}{k}^{m_i} \right) \sum_{l=0}^{\sum_{i=1}^s n_i m_i - k} \sum_{i=1}^s m_i \binom{\sum_{i=1}^s n_i m_i - k}{l} (-1)^l \\
&\quad \times \int_{\mathbb{Z}_p} [x]_q^{(m_1 + \dots + m_s)k + l} d\mu_q(x) \\
&= \left( \prod_{i=1}^s \binom{n_i}{k}^{m_i} \right) \sum_{l=0}^{\sum_{i=1}^s n_i m_i - k} \sum_{i=1}^s m_i \binom{\sum_{i=1}^s n_i m_i - k}{l} (-1)^l \\
&\quad \times \beta_{(m_1 + \dots + m_s)k + l, q}.
\end{aligned} \tag{19}$$

By (18) and (19), we obtain the following theorem.

**Theorem 6.** For  $s \in \mathbb{N}$  and  $m_1, \dots, m_s, n_1, \dots, n_s, k \in \mathbb{Z}_+$  with  $m_1 n_1 + \dots + m_s n_s > (m_1 + \dots + m_s)k + 1$ , we have

$$\begin{aligned}
& \sum_{l=0}^{k \sum_{i=1}^s m_i} \binom{k \sum_{i=1}^s m_i}{l} (-1)^{k \sum_{i=1}^s m_i - l} \left\{ \left( \sum_{i=1}^s m_i n_i - l + 1 \right) - q + q^2 \beta_{n_1 m_1 + n_s m_s - l, \frac{1}{q}} \right\} \\
&= \sum_{l=0}^{\sum_{i=1}^s n_i m_i - k \sum_{i=1}^s m_i} \binom{\sum_{i=1}^s n_i m_i - k \sum_{i=1}^s m_i}{l} (-1)^l \beta_{(m_1 + \dots + m_s)k + l, q}.
\end{aligned}$$

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